

- [7] R.A.Johnson (1994) *Miller & Freund's Probability and Statistics for Engineers, 5th edition*, Prentice Hall, New Jersey 07632.
- [8] J.Carroll (2002) *Human - Computer Interaction in the New Millennium*, ACM Press, NY.
- [9] A.M.Law and W.D.Kelton (2000) *Simulation Modeling and Analysis*, 3d edition, McGraw-Hill, N.Y.

Author Information

B. Dimitrov - bdimitro@kettering.edu

D. Green, Jr. - dgreen@kettering.edu

V.Rykov - vrykov@kettering.edu

P. Stanchev - pstanchev@kettering.edu

Department of Science & Mathematics, Kettering University, Flint, Michigan, USA

1700 West Third Avenue, Flint, Michigan 48504-4898, USA

A GRADIENT-TYPE OPTIMIZATION TECHNIQUE FOR THE OPTIMAL CONTROL FOR SCHRÖDINGER EQUATIONS

M. H. FARAG

Abstract: In this paper, we are considered with the optimal control of a schrodinger equation. Based on the formulation for the variation of the cost functional, a gradient-type optimization technique utilizing the finite difference method is then developed to solve the constrained optimization problem. Finally, a numerical example is given and the results show that the method of solution is robust.

Keywords: Optimal control, schrodinger equation, Existence and uniqueness theory, Gradient method.

AMS subject classification: 49J20, 49M29, 49M30, 49K20

1. Introduction

Optimal control of systems governed by partial differential equations is an application-driven area of mathematics involving the formulation and solution of minimization problems [1,3]. In this paper, we are considered with the optimal control of a schrodinger equation. Based on the formulation for the variation of the cost functional, a gradient-type optimization technique utilizing the finite difference method is then developed to solve the constrained optimization problem. Finally, a numerical example is given and the results show that the method of solution is robust.

2. Problem Formulation

We consider the functional on the form

$$(1) \quad J(u) = \alpha_0 \int_0^T |y(0,t) - f_0(t)|^2 dt + \alpha_1 \int_0^T |y(1,t) - f_1(t)|^2 dt$$

which is to be minimized under the conditions

$$(2) \quad i \frac{\partial y}{\partial t} + B_0 \frac{\partial^2 y}{\partial x^2} - u y = f(x,t), \quad (x,t) \in \Omega = (0,1) \times (0,T)$$

$$(3) \quad y(x,0) = 0, \quad x \in (0,1)$$

$$(4) \quad \frac{\partial y(0,t)}{\partial x} = \frac{\partial y(1,t)}{\partial x} = 0, \quad t \in (0,T)$$

over the class

$$U = \left\{ u : u(x,t) \in W_2^{0,1}(\Omega), \alpha_0 \leq u(x,t) \leq \alpha_1, |u_t| \leq \alpha_2, \forall (x,t) \in \Omega \right\}$$

where $\alpha_k \geq 0, k = \overline{0,2}, \alpha_1 + \alpha_2 \neq 0, 1, T, B_0 > 0$ are given numbers

and $f_0(t), f_1(t) \in W_2^1(0, T), \varphi(x) \in W_2^1(0, 1)$, are given functions.

Definition 1.

The problem of finding the function $y(x, t) \in V_2^{0,1}(\Omega)$ from condition (2)-(4) at given $u \in U$ is called the reduced problem.

Definition 2.

A function $y(x, t) \in V_2^{0,1}(\Omega)$ is said to be a solution of the problem (2)-(4), if for all $\eta = \eta(x, t) \in W_2^{1,1}(\Omega)$ the equation

$$(5) \int_{\Omega} \left[-i y \frac{\partial \eta}{\partial t} - B_0 \frac{\partial y}{\partial x} \frac{\partial \eta}{\partial x} - u y \bar{\eta} \right] dx dt = \int_{\Omega} f(x, t) \bar{\eta} dx dt + i \int_0^1 \varphi \bar{\eta}(x, 0) dx$$

is valid and $\eta(x, T) = 0$, but $\bar{\eta}$ is the adjoint of η .

Proposition 1

Let $f(x, t) \in W_2^{0,1}(\Omega)$ and $\varphi(x, t) \in W_2^1(0, 1)$. Then the problem (2)-(4) has a unique solution and satisfies the following estimate

$$(6) \|y\|_{V_2^{1,0}(\Omega)}^2 \leq C_1 [\|\varphi\|_{W_2^{1,0}(0,1)}^2 + \|f\|_{W_2^{0,1}(0,1)}^2] \text{ is valid and } C_1 > 0 \text{ does not depend on } \varphi \text{ and } f.$$

Proposition 2

Let $\varphi(x, t) \in W_2^2(0, 1)$. Then the solution of the reduced problem (2)-(4) $y(x, t) \in V_2^{0,1}(\Omega)$

belongs to the space $W_2^{2,1}(\Omega)$ and satisfies the following estimate

$$(7) \|y\|_{W_2^2(\Omega)}^2 + \|y_t\|_{L_2(0,1)}^2 \leq C_2 [\|\varphi\|_{W_2^2(0,1)}^2 + \|f\|_{W_2^{0,1}(\Omega)}^2] \text{ is valid and}$$

$\forall t \in [0, T], C_2 > 0$ does not depend on φ and f .

Proposition 3

Let all the conditions Proposition 2 be valid. Then the optimal control problem (1)-(4) has at least one solution.

3. Variation of the Cost Functional

3.1 The Adjoint Problem

Results [4] imply that the function $\Phi = \Phi(x, t, u)$ is a solution in $L_2(\Omega)$ of the adjoint problem

$$(8) \quad i \frac{\partial \Phi}{\partial t} + B_0 \frac{\partial^2 \Phi}{\partial x^2} - u \Phi = 0, (x, t) \in \Omega = (0, 1) \times (0, T)$$

$$\begin{aligned}
 & \Phi(x, T) = 0, \quad x \in (0, 1) \\
 (9) \quad & \frac{\partial \Phi(0, t)}{\partial x} = -\frac{2\alpha_0}{B_0} [y(0, t) - f_0(t)], \quad t \in (0, T) \\
 & \frac{\partial \Phi(1, t)}{\partial x} = \frac{2\alpha_1}{B_0} [y(1, t) - f_1(t)], \quad t \in (0, T)
 \end{aligned}$$

where $y(x, t)$ is the solution of (1)-(4) corresponding to $u \in U$.

Definition 3.

For each $u \in U$, a function $\Phi(x, t; u)$ is a solution of the adjoint problem (8)-(9) belonging to the control U iff

(I) $\Phi(x, t; u) \in L_2(\Omega)$,

(II) The integral identity

$$\begin{aligned}
 (10) \quad & \int_{\Omega} \Phi \left[i \frac{\partial \overline{\eta_1}}{\partial t} + B_0 \frac{\partial^2 \overline{\eta_1}}{\partial x^2} - u \overline{\eta_1} \right] dx \, dt \\
 & = -2\alpha_1 \int_0^T [y(1, t) - f_1(t)] \overline{\eta_1}(1, t) \, dt \\
 & + 2\alpha_0 \int_0^T [y(0, t) - f_0(t)] \overline{\eta_1}(0, t) \, dt
 \end{aligned}$$

is valid $\forall \eta_1 \in W_{2,1}^{2,1}(\Omega)$, $\eta_1(x, 0) = (\eta_1)_x|_{x=0} = (\eta_1)_x|_{x=1} = 0$.

On the basis of the above assumptions and the results [5], we have the following proposition:

Proposition 4.

The adjoint problem (8)-(9) has a unique solution from $L_2(\Omega)$ and the following estimate

$$\begin{aligned}
 (11) \quad & \|\Phi\|_{L_2(\Omega)}^2 \leq C_3 [\Gamma_1 + \Gamma_2], \quad \text{where} \\
 & \Gamma_1 = \|y(0, t) - f_0(t)\|_{W_2^{\frac{1}{2}}(0, T)}^2, \quad \Gamma_2 = \|y(1, t) - f_1(t)\|_{W_2^{\frac{1}{2}}(0, T)}^2
 \end{aligned}$$

is valid and C_3 is a certain constant.

3.2 The Gradient Formulae of Cost Functional

The sufficient differentiability conditions of the functional (5) and its gradient formulae will be given as follows:

Theorem 1.

Let the above assumptions be satisfied. Then $J(u)$ is Gato differentiable, and its gradient satisfies

$$(12) \quad \delta J(u) = - \int_{\Omega} \operatorname{Re}(y \Phi) \omega \, dx \, dt, \quad \forall \omega \in W_{\infty}^{0,1}(\Omega).$$

Proof:

Suppose that $u \in U$ and $\delta u \in W_{\infty}^{0,1}(\Omega)$ such that $u + \delta u \in U$ and denoting $\delta y(x, t) = y(x, t; u + \delta u) - y(x, t; u)$. Then $\delta y(x, t; \delta u)$ is the solution of the boundary value problem:

$$(13) \quad i \frac{\partial \delta y}{\partial t} + B_0 \frac{\partial^2 \delta y}{\partial x^2} - (u + \delta u) \delta y = y(x, t) \delta u, (x, t) \in \Omega,$$

$$(14) \quad \delta y(x, 0) = 0, x \in (0, 1), \frac{\partial \delta y(0, t)}{\partial x} = \frac{\partial \delta y(1, t)}{\partial x} = 0, t \in (0, T)$$

and the solution of the above boundary value problem satisfies the following estimation

$$(15) \quad \|\delta y\|_{W_2^{2,0}(\Omega)}^2 \leq C_4 \|\delta u y\|_{W_2^{0,1}(\Omega)}^2$$

where C_4 is a constant and independent of δu .

From (15) and using the theorem of imbedding [6], we have

$$(16) \quad \|\delta y(0, t)\|_{L_2(0, T)} + \|\delta y(1, t)\|_{L_2(0, T)} \leq C_5 \|\delta u y(x, t)\|_{W_2^{0,1}(\Omega)}$$

where C_5 is a constant and independent of δu .

The increment of the functional $J(u)$ can be expressed as:

$$(17) \quad \begin{aligned} \delta J &= J(u + \theta u) - J(u) \\ &= 2 \alpha_1 \operatorname{Re} \int_0^T [y(1, t) - f_1(t)] \overline{\delta y(1, t)} dt \\ &\quad + 2 \alpha_0 \operatorname{Re} \int_0^T [y(0, t) - f_0(t)] \overline{\delta y(0, t)} dt \\ &\quad + \alpha_1 \|\delta y(1, t)\|_{L_2(0, T)}^2 + 2 \alpha_0 \|\delta y(0, t)\|_{L_2(0, T)}^2 \end{aligned}$$

If we take complex adjoint for (10), (13), we have

$$(18) \quad \begin{aligned} \int_{\Omega} \overline{\Phi} \left[i \frac{\partial \eta_1}{\partial t} + B_0 \frac{\partial^2 \eta_1}{\partial x^2} - u \eta_1 \right] dx dt \\ = -2 \alpha_1 \int_0^T [\overline{y(1, t)} - \overline{f_1(t)}] \eta_1(1, t) dt \\ + 2 \alpha_0 \int_0^T [\overline{y(0, t)} - \overline{f_0(t)}] \eta_1(0, t) dt \end{aligned}$$

$$(19) \quad \begin{aligned} \int_{\Omega} \left[i \frac{\partial \overline{\delta y}}{\partial t} + B_0 \frac{\partial^2 \overline{\delta y}}{\partial x^2} - (u + \delta u) \overline{\delta y} \right] \eta dx dt \\ = \int_{\Omega} \overline{y(x, t)} \delta u \eta dx dt, \end{aligned}$$

Subtracting (13) from (19), (10) from (18) and in the obtained relation we put $\Phi, \delta y$ instead of η, η_1 , then we have

$$\begin{aligned}
 (20) \quad & 2 \alpha_1 \operatorname{Re} \int_0^T [y(1, t) - f_1(t)] \overline{\delta y}(1, t) dt \\
 & + 2 \alpha_0 \operatorname{Re} \int_0^T [y(0, t) - f_0(t)] \overline{\delta y}(0, t) dt \\
 & = -\frac{1}{2} \int_{\Omega} [\delta u \Phi \bar{y} + \delta u y \bar{\Phi}] dx dt \\
 & \quad - \frac{1}{2} \int_{\Omega} [\delta u \delta y \bar{\Phi} + \delta u \Phi \bar{\delta y}] dx dt \\
 & = -\operatorname{Re} \int_{\Omega} y \bar{\Phi} \delta u dx dt - \operatorname{Re} \int_{\Omega} \delta y \bar{\Phi} \delta u dx dt.
 \end{aligned}$$

By substituting the last relation in (17), we have

$$\begin{aligned}
 (21) \quad \delta J = & -\operatorname{Re} \int_{\Omega} y \bar{\Phi} \delta y \delta u dx dt - \operatorname{Re} \int_{\Omega} \delta y \bar{\Phi} \delta u dx dt. \\
 & + \alpha_0 \|\delta y(0, t)\|_{L_2(0, T)}^2 + \alpha_1 \|\delta y(1, t)\|_{L_2(0, T)}^2
 \end{aligned}$$

Suppose that

$$(22) \quad R_1 = \alpha_0 \|\delta y(0, t)\|_{L_2(0, T)}^2 + \alpha_1 \|\delta y(1, t)\|_{L_2(0, T)}^2$$

$$(23) \quad R_2 = -\operatorname{Re} \int_{\Omega} \delta y \bar{\Phi} \delta u dx dt.$$

It is clear that,

$$(24) \quad |R_1| \leq \alpha_0 \|\delta y(0, t)\|_{L_2(0, T)}^2 + \alpha_1 \|\delta y(1, t)\|_{L_2(0, T)}^2.$$

From the formulae of R_2 , it is estimated as

$$(25) \quad |R_2| \leq C \|\delta y \delta u\|_{L_2(\Omega)}^2.$$

Then

$$(26) \quad |R_1| + |R_2| = o(\|\delta u\|_{W_{\infty}^{0,1}(\Omega)}).$$

By substituting (26) in (21), we obtain

$$(27) \quad J(u + \theta \delta u) - J(u) = -\int_{\Omega} \operatorname{Re}(y \bar{\Phi}) (\theta \omega) dx dt + O(\theta).$$

Hence, in light of the variation functional, we have

$$(28) \quad \delta J(u, \omega) = \lim_{\theta \rightarrow 0} \frac{J(u + \theta \delta u) - J(u)}{\theta} = -\int_{\Omega} \operatorname{Re}(y \bar{\Phi}) \omega dx dt$$

and this proves the differentiability of the functional and gradient formulae of the function $J(u)$. This completes the proof of the theorem.

Using Tikhinov method [7], we define the following functional

$$(29) \quad J_m(u) = J(u) + \alpha^m \int_0^1 \int_0^T |u(x, t) - \omega(x, t)|^2 dx dt.$$

and $\omega(x, t) \in L_2(\Omega)$.

4. Discrete Problem

We consider the set of node values $\{x_j, t_k\}$, $x_j = x_0 + jh$, $j = \overline{0, M}$

$t_k = t_0 + k\tau$, $k = \overline{0, N}$, $M = \frac{1}{h}$, $N = \frac{T}{\tau}$ and the following notations [8]:

$$(30) \quad (y_j^k)_x = \frac{y_j^k - y_{j-1}^k}{h}, \quad (y_j^k)_x = \frac{y_{j+1}^k - y_j^k}{h},$$

$$(31) \quad (y_j^k)_t = \frac{y_j^k - y_j^{k-1}}{\tau}, \quad (y_j^k)_{xx} = \frac{y_{j+1}^k - 2y_j^k + y_{j-1}^k}{h^2}$$

After applying the numerical integration formula [8], we have the discretisation of the optimal control problem (1)-(5) as follows: Let it is required to minimize the functional

$$(32) \quad I_m([u]) = \tau \sum_{k=0}^N \{ \alpha_0 [y_0^k - f_0^k]^2 - \alpha_1 [y_M^k - f_1^k]^2 \} \\ + v^m \tau \sum_{k=1}^N \{ h \sum_{j=1}^{M-1} |u_0^k - \omega_0^k|^2 + \frac{1}{2} |u_M^k - \omega_M^k|^2 + \frac{1}{2} |u_j^k - \omega_j^k|^2 \}$$

on the control set

$$U_N^M = \left\{ [u] : [u] = (u_j^k), \alpha_0 \leq u_j^k \leq \alpha_1, j = \overline{0, M}, k = \overline{0, N}, \right. \\ \left. | (u_j^k)_t | \leq \alpha_2, j = \overline{0, M}, k = \overline{2, N} \right\}$$

under the conditions

$$(33) \quad i (y_j^k)_t + B_0 (y_j^k)_{xx} - u_j^k y_j^k = f_j^k, j = \overline{1, M-1}, k = \overline{1, N}$$

$$(34) \quad y_j^0 = 0, j = \overline{0, M},$$

$$(35) \quad \frac{2B_0}{h} (y_0^k)_x = f_0^k - i [(u_0^k)_t - u_0^k y_0^k], k = \overline{1, N}$$

$$(36) \quad -\frac{2B_0}{h} (y_M^k)_x = f_M^k - i [(u_M^k)_t - u_M^k y_M^k], k = \overline{1, N}$$

Now, the discrete gradient formulae will be given as follows:

Theorem 2

The functional $J_m(u)$ is differentiable, and its gradient satisfies

$$(37) \quad (I'_m([u]))_j^k = -\operatorname{Re} (y_j^k \Phi_j^k) + 2\alpha^m (u_j^k - \omega_j^k),$$

where $j = \overline{0, M-1}, k = \overline{1, N}$ and Φ_j^k is the solution of discrete adjoint problem:

$$(38) \quad i (\Phi_j^k)_t + B_0 (\Phi_j^k)_{xx} - u_j^k \Phi_j^k = 0, j = \overline{1, M-1}, k = \overline{1, N-1}$$

$$(39) \quad \Phi_j^N = 0, j = \overline{0, M},$$

$$(40) \quad (\Phi_j^k)_x + \frac{2\alpha_0}{B_0} [y_0^k - f_0^k] = \frac{h}{B_0} [u_0^k \Phi_0^k - i (\Phi_0^k)_t], k = \overline{1, N-1}$$

$$(41) (\Phi_j^k)_x + \frac{2\alpha_0}{B_0} [y_M^k - f_1^k] = \frac{h}{B_0} [u_M^k \Phi_M^k - i (\Phi_M^k)_t], k=\overline{1, N-1}$$

5. Solution of Control Problem

5.1 The Projection Gradient Method

Here we describe the projection gradient method [9] for the solution of the optimal control problem such as: construct a sequence u_{n+1m} by setting

$$(42) [u]_{n+1m} = P_{U_N^M} \left\{ [u]_{nm} - v_n (I_m^1 ([u]_{nm})) \right\}$$

where $P_{U_N^M}(u)$ is the project on the set U_N^M . In the first we define $(\bar{u}_j^k)_{nm}$ in the form

$$(43) (\bar{u}_j^k)_{n+1m} = \begin{cases} \Psi_1 & \alpha_0 \leq \Psi_2 \leq \alpha_1 \\ \alpha_0 & \Psi_2 < \alpha_0 \\ \alpha_1 & \Psi_2 > \alpha_1 \end{cases}$$

where

$$(44) \Psi_2 = [u_j^k]_{nm} - v_n (I_m^1 ([u]_{nm}))_j^k$$

$$(45) \Psi_1 = [u_j^k]_{nm} + v_n (I_m^1 ([u]_{nm}))_j^k$$

and $j=\overline{0, M}$, $k=\overline{1, N}$, $n=0, 1, \dots$, $m=0, 1, \dots$

Using the above sequence we construct the project in the form

$$(46) (u_j^1)_{n+1m} = (\bar{u}_j^1)_{n+1m}$$

$$(47) (u_j^k)_{n+1m} = \begin{cases} \Theta_0 & \Theta_1 \leq \Psi_2 \leq \Theta_2 \\ \Theta_1 & (u_j^k)_{nm} < \Theta_1 \\ \Theta_2 & (u_j^k)_{nm} > \Theta_2 \end{cases}$$

where

$$\Theta_0 = (\bar{u}_j^k)_{n+1m} + v_n (I_m^1 ([u]_{nm}))_j^k$$

$$\Theta_1 = -\tau \alpha_2 + (\bar{u}_j^{k-1})_{n+1m}, \quad \Theta_2 = \tau \alpha_2 + (\bar{u}_j^{k-1})_{n+1m}$$

$j=\overline{0, M}$, $k=\overline{2, N}$, $n=0, 1, \dots$, $m=0, 1, \dots$

5.2 Numerical Algorithm

With the gradient obtained, the following gradient type algorithm can then be developed for the optimal value of u^* based on the projection gradient method (PGM) which described in the above section.

The outlined of the algorithm for solving control problem are as follows:

Step 1: Choose an initial control $u^{(n)} \in U$, $n=0$.

If $I'(u^{(n)})=0$, $u^{(n)}$ is the solution of the problem.

Step 2 : At each iteration n do

Solve the state problem, then find $y(\cdot, u^{(n)})$.

Solve the adjoint problem for (1)-(3), then find

$\Phi(\cdot, u^{(n)})$. Find optimal control $u_*^{(n+1)}$ using PGM.

End do.

Step 3: Test the optimality of $u^{(n+1)}$.

If $u^{(n+1)}$ is optimum, stop the process.

Otherwise, go to Step 4.

Step 4 Set $u^{(n+1)} = u^{(n)}$, $n = n + 1$ and go to Step 2.

6. Numerical Results

Designed algorithm is implemented as a FORTRAN routine [10]. Numerical experiment is carried out to check its performance. The initial data of the problem (1)-(5) are taken as follows:

$$\alpha_0 = \alpha_1 = \alpha_2 = 1 = T = 1, \varepsilon = 0.5E-03$$

$$f_0 = it, f_1 = i(1+t), \varphi(x) = ix, u^{(0)} = 1.0$$

$$\omega(x,t) = 1 + \frac{x+t}{2}, f(x,t) = -1 - i(x+t)(x^2+t+1)$$

The number of division of the intervals was taken as $N=M=20$. The computed control values of

$u_j^{13}, j = \overline{0, N}$ the values of relative error are shown in Tables 1, 2 and the 3D plots of the optimal control and initial values are presented in Figures 1, 2. The optimal value of the cost functional is

$$J^* = \inf_{u \in U} J(u) = J(u^*) = 0.48526E-03.$$

The computed control values of $u_j^{13}, j = \overline{0, N}$			
0.15592E+01	0.15950E+01	0.16301E+01	0.16641E+01
0.17221E+01	0.17464E+01	0.17714E+01	0.18021E+01
0.18332E+01	0.18602E+01	0.19112E+01	0.19830E+01
0.20679E+01	0.21474E+01	0.22155E+01	0.22837E+01
0.23625E+01	0.24368E+01	0.24078E+01	0.24324E+01
0.24718E+01			

The values of relative error of $u_j^{13}, j = \overline{0, N}$			
0.025528	0.004683	0.012478	0.02563
0.050038	0.030459	0.048194	0.046236
0.041988	0.032024	0.033075	0.042328
0.055069	0.061758	0.060028	0.056052
0.054668	0.049211	0.000924	0.028029
0.049297			

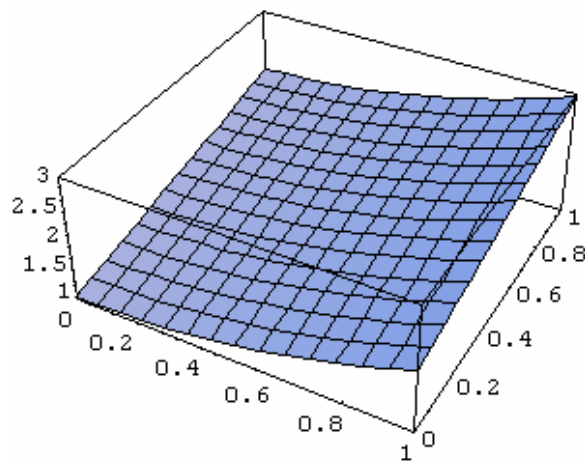


Fig. 1. Optimal control $u_*(x,t)$

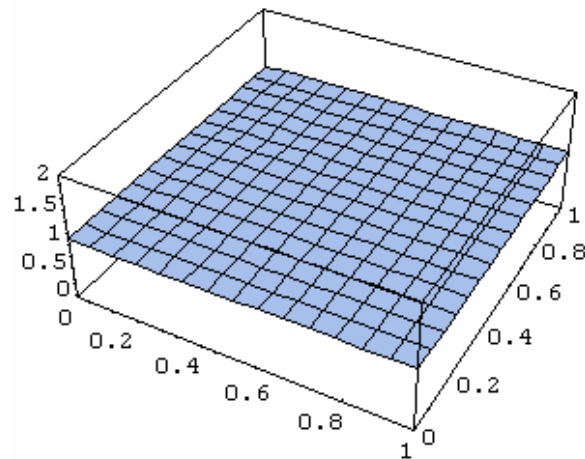


Fig. 2. Initial control $u_0(x,t)$

REFERENCES

- [1] A. J. Kearsley, The use of optimization techniques in the solution of partial differential equations from science and engineering, PhD, Houston, Texas, 1996.
- [2] K. Kime, Finite difference approximation of control via the potential in a 1-D Schrodinger equation, Journal of Differential equations, 2000(26), 1--10(2000).
- [3] N. U. Ahmed and K. L. Teo, Optimal control of distributed parameter systems, North Holland New York, Oxford(1981).
- [4] S. H. Farag and M. H. Farag, Necessary optimality conditions in control problems for hyperbolic equations, Journal Egypt. Math. Soc., 8(2), 1--10(2000).
- [5] O. A. Ladyzhenskaya, Boundary value problems of mathematical physics, Nauka, Moscow, Russian(1973).
- [6] V. P. Mikhailov, Partial differential equations, Nauka, Moscow, Russian(1983).
- [7] A. N. Tikhonov and N. Ya. Arsenin, Methods for the solution of incorrectly posed problems, Nauka, Moscow, Russian(1974).
- [8] A. H. Khater, A. B. Shamardan, M. H. Farag and A. H. Abel-Hamid, Analytical and numerical solutions of a quasilinear parabolic optimal control problem, Journal Comput. and Appl. Math., 95, 29--43(1998).
- [9] J. T. Betts, Survey of numerical methods for trajectory optimization, Journal of Guidance control and dynamics, 21(2), 193--207(1998).
- [10] D. Kraft, Algorithm 773: TOMP-Fortran modules for optimal control calculations, ACM Transactions of mathematical software, 20(2), 262--281(1994).

Author Information

M. H. FARAG : Department of Mathematics and Computer Science; Faculty of Education, Ibri, Sultanate of Oman. or: Faculty of Science, El-Minia University, Egypt. E-mail: farag5358@yahoo.com